Cover Schemes, Frame-Valued Sets and Their Potential Uses in Spacetime Physics

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Abstract

In the present paper the concept of a covering is presented and developed. The relationship between cover schemes, frames (complete Heyting algebras), Kripke models, and frame-valued set theory is discussed. Finally cover schemes and frame-valued set theory are applied in the context of Markopoulou's account of discrete spacetime as sets "evolving" over a causal set. We observe that Markopoulou's proposal may be effectively realized by working within an appropriate frame-valued model of set theory. We go on to show that, within this framework, cover schemes may be used to force certain conditions to prevail in the associated models: for example, rendering the universe timeless, obliterating a given event or forcing it to become the universe's "beginning".

Preamble

The concept of *Grothendieck (pre)topology* or *covering* issued from the efforts of algebraic geometers to study "sheaf-like" objects defined on categories more general than the lattice of open sets of a topological space (see, e.g. [4]). A *Grothendieck pretopology* on a category $\mathscr{C}$ with pullbacks is defined by specifying, for each object U of $\mathscr{C}$, a set $P(U)$ of arrows to U called *covering families* satisfying appropriate category theoretic versions of the corresponding conditions for a family $\mathscr{A}$ of sets to cover a set U, namely: (i) $\{U\}$ covers U, (ii) if $\mathscr{A}$ covers U and $V \subseteq U$, then $\mathscr{A}|V = \{A \cap V: A \in \mathscr{A}\}$ covers V, and (iii) if $\mathscr{A}$ covers U and, for each $A \in \mathscr{A}$, $\mathscr{B}_A$ covers A, then $\bigcup_{A \in \mathscr{A}} \mathscr{B}_A$ covers U. In the present paper the covering concept—here called a *cover scheme*—is presented and developed in the simple case when the underlying category is a preordered set. The relationship between cover schemes, frames (complete Heyting algebras), Kripke models, and frame-valued set theory is discussed. Finally cover schemes and frame-valued set theory are applied in the context of Markopoulou's [5] account of discrete spacetime as sets "evolving" over a causal set.

I. COVER SCHEMES ON PREORDERED SETS

A *preordered set* is a set equipped with a reflexive transitive relation $\leq$. Let $(P, \leq)$ be a fixed but arbitrary preordered set: we shall use letters p, q, r, s, t to denote elements of p. We write $p \cong q$ for $(p \leq q \ \&\ q \leq p)$. A *meet* for a subset S of P is an element p of P such that $\forall q[\forall s \in S(q \leq s) \leftrightarrow q \leq p]$: if p and p' are both meets for S, then $p \cong p'$. If the empty subset $\varnothing$ has a meet, any such meet m is necessarily a *largest* or *top* element of P, that is, satisfies $p \leq m$ for all p. We use the symbol 1 to denote a top element of P. A meet of a finite subset $\{p_1, \ldots, p_n\}$ of P will be denoted by $p_1 \wedge \ldots \wedge p_n$. P is a *lower semilattice* if each nonempty finite subset of P has a meet. A subset S of P is said to be a *sharpening of,* or to *sharpen,* a subset T of P, written $S \prec T$, if $\forall s \in S \exists t \in T(s \leq t)$. A *sieve* in P is a subset S such that $p \in S$ and $q \leq p$ implies $q \in S$. Each subset S of P generates a sieve $\overline{S}$ given by $\overline{S} = \{p : \exists s \in S(p \leq s)\}$.

A *cover scheme* on P is a map $\mathbf{C}$ assigning to each $p \in P$ a family $\mathbf{C}(p)$ of subsets of $p\downarrow = \{q\colon q \le p\}$, called ($\mathbf{C}$-)*covers of* p, such that, if $q \le p$, any cover of p can be sharpened to a cover of q, i.e.,

(Cov) $$S \in \mathbf{C}(p) \,\&\, q \le p \to \exists T \in \mathbf{C}(q)[\forall t \in T \exists s \in S(t \le s)].$$

If P is a lower semilattice, a *coverage* (see [3]) on P is a map $\mathbf{C}$ as above, satisfying, in place of **(Cov)**, the condition

$$S \in \mathbf{C}(p) \,\&\, q \le p \to S \wedge q = \{s \wedge q : s \in S\} \in \mathbf{C}(p).$$

A cover scheme $\mathbf{C}$ is said to be *normal* if every member of every $\mathbf{C}(p)$ is a sieve and whenever $S \in \mathbf{C}(p)$ and T is a sieve such that $S \subseteq T \subseteq p\downarrow$, we have $T \in \mathbf{C}(p)$. Any cover scheme $\mathbf{C}$ on P induces a normal cover scheme $\overline{\mathbf{C}}$ (called its *normalization*) defined by

$$\overline{\mathbf{C}}(p) = \{X \subseteq p\downarrow : X \text{ is a sieve } \& \exists S \in \mathbf{C}(p).\ S \subseteq X\}.$$

Notice that a normal cover scheme on a lower semilattice is always a coverage. For if $\mathbf{C}$ is such, then for $S \in \mathbf{C}(p)$ and $q \le p$, any sharpening of S to a member of $\mathbf{C}(q)$ is easily seen to be included in $S \wedge q$, so that the latter is also in $\mathbf{C}(q)$.

Write $\mathcal{Cov}(P)$ for the set of all cover schemes on P. There is a natural partial ordering $\lhd$ on $\mathcal{Cov}(P)$ defined by

$$\mathbf{C} \lhd \mathbf{D} \leftrightarrow \forall p\ \mathbf{C}(p) \subseteq \mathbf{D}(p).$$

With this ordering $\mathcal{Cov}(P)$ is a complete lattice in which the join $\bigvee_{i \in I} \mathbf{C}_i$ of any family $\{\mathbf{C}_i\colon i \in I\}$ is given by

$$(\bigvee_{i \in I} \mathbf{C}_i)(p) = \bigcup_{i \in I} \mathbf{C}_i(p).$$

There is also a natural *composition* ★ defined on $\mathcal{Cov}(P)$. For $\mathbf{C}, \mathbf{D} \in \mathcal{Cov}(P)$, $\mathbf{D}$★$\mathbf{C}$ is defined by decreeing that ($\mathbf{D}$★$\mathbf{C}$)(p) is to consist of all subsets of $p\downarrow$ of the form $\bigcup_{s \in S} T_s$, where $S \in \mathbf{C}(p)$ and, for each $s \in S$, $T_s \in \mathbf{D}(s)$. That $\mathbf{D}$★$\mathbf{C}$ is a cover scheme on P may be verified (using the axiom of choice) as follows. Given $S \in \mathbf{C}(p)$, $\bigcup_{s\in S} T_s \in (\mathbf{D}\bigstar\mathbf{C})(p)$ and $q \le p$, there is $U \in \mathbf{C}(q)$ with $U \prec S$, so for each $u \in U$ there is $s(u) \in S$ for which $u \le s(u)$. Then $T_{s(u)} \in \mathbf{D}(s(u))$ and we can choose $V_u \in \mathbf{D}(u)$ so that $V_u \prec T_{s(u)}$. Clearly $\bigcup_{u\in U} V_u \in (\mathbf{D}\bigstar\mathbf{C})(q)$ and, since $V_u \prec T_{s(u)}$ for all $u \in U$, it follows immediately that $\bigcup_{u\in U} V_u \prec \bigcup_{s\in S} T_s$.

It is not hard to verify that $\bigstar$ is associative and that with this operation $\mathscr{Cov}(P)$ is actually a *quantale* (see, e.g., [6]) that is, for any **D,** $\{\mathbf{C}_i: i \in I\}$ in $\mathscr{Cov}(P)$,

$$\mathbf{D}\bigstar\bigvee_{i\in I}\mathbf{C}_i = \bigvee_{i\in I}(\mathbf{D}\bigstar\mathbf{C}_i) \qquad (\bigvee_{i\in I}\mathbf{C}_i)\bigstar\mathbf{D} = \bigvee_{i\in I}(\mathbf{C}_i\bigstar\mathbf{D}) \quad .$$

Also the element $\mathbf{1} \in \mathscr{Cov}(P)$ with $\mathbf{1}(p) = \{p\}$ acts as a quantal unit, since it is readily verified that $\mathbf{1}\bigstar\mathbf{C} = \mathbf{C}\bigstar\mathbf{1} = \mathbf{C}$ for all $\mathbf{C} \in \mathscr{Cov}(P)$.

In this connection a *Grothendieck pretopology*—which we shall abbreviate simply to *pretopology*—on P may be identified as a cover scheme $\mathbf{C}$ on P satisfying $\mathbf{1} \lhd \mathbf{C}$ and $\mathbf{C}\bigstar\mathbf{C} \lhd \mathbf{C}$, that is, $\{p\} \in \mathbf{C}(p)$ for all $p \in P$ and, if $S \in \mathbf{C}(p)$ and, for each $s \in S$, $T_s \in \mathbf{C}(s)$, then $\bigcup_{s\in S} T_s \in \mathbf{C}(p)$.

We observe that a normal pretopology $\mathbf{C}$ has the additional properties: (i) each $\mathbf{C}(p)$ is a filter of sieves in $p\!\downarrow$, that is, satisfies $S, T \in \mathbf{C}(p) \leftrightarrow S \in \mathbf{C}(p)\ \&\ T \in \mathbf{C}(p)$; (ii) $S \in \mathbf{C}(p)\ \&\ q \le p \rightarrow S \cap q\!\downarrow \in \mathbf{C}(q)$. For (ii), we observe that $S \cap q\!\downarrow$, including as it does any sharpening of S to a member of $\mathbf{C}(q)$, is itself a member of $\mathbf{C}(q)$. As for (i), the "$\rightarrow$" direction is obvious; conversely, if $S, T \in \mathbf{C}(p)$, then $S \cap T = S \cap \bigcup_{t\in T}(t\!\downarrow) = \bigcup_{t\in T}(S \cap t\!\downarrow)$. But from (ii) we have $S \cap t\!\downarrow \in \mathbf{C}(t)$ for every $t \in T$, whence $\bigcup_{t\in T}(S \cap t\!\downarrow) \in \mathbf{C}(p)$, and so $S \cap T \in \mathbf{C}(p)$.

A normal pretopology is also called a *Grothendieck topology.* A normal cover scheme satisfying (i) and (ii) is called a *regular* cover scheme.

Each cover scheme **C** generates a pretopology, and a Grothendieck topology in the following way. First, define $\mathbf{C}^n$ for $n \in \omega$ recursively by $\mathbf{C}^0 = \mathbf{1}$ and $\mathbf{C}^{n+1} = \mathbf{C}\bigstar\mathbf{C}^n$. Now put $\mathbf{G} = \bigvee_{n\in\omega} \mathbf{C}^n$. Then **G** is a pretopology, for obviously $\mathbf{1} \lhd \mathbf{G}$, and

$$\mathbf{G} \bigstar \mathbf{G} = \bigvee_{n\in\omega} \mathbf{C}^n \bigstar \bigvee_{m\in\omega} \mathbf{C}^m = \bigvee_{n\in\omega} \bigvee_{m\in\omega} \mathbf{C}^{m+n} = \bigvee_{n\in\omega} \mathbf{C}^n = \mathbf{G}.$$

Also $\mathbf{C} \lhd \mathbf{G}$, and **G** is evidently the $\lhd$-least such pretopology. **G** is called the *pretopology generated by* **M.** The normalization $\overline{\mathbf{G}}$ of **G** is then a Grothendieck topology called the *Grothendieck topology generated by* **C.**

Now let **M** be a map assigning to each $p \in P$ a subset $\mathbf{M}(p)$ of subsets of $p\downarrow$. Since $\mathcal{Cov}(P)$ is a complete lattice, there is a $\lhd$-least cover scheme **C** such that $\mathbf{M}(p) \subseteq \mathbf{C}(p)$ for all p. **C** is called the cover scheme *generated by* **M**; the pretopology and Grothendieck topology generated in turn by **C** are said to be *generated by* **M.**

There are several naturally defined cover schemes on P which also happen to be pretopologies. First, each sieve A in P determines two cover schemes $\mathbf{C}_A$ and $\mathbf{C}^A$ defined by

$$S \in \mathbf{C}_A(p) \leftrightarrow p \in A \cup S \qquad S \in \mathbf{C}^{\mathbf{A}}(p) \leftrightarrow p\downarrow \cap A \subseteq S:$$

these are easily shown to be pretopologies. Notice that $\varnothing \in \mathbf{C}_A(p) \leftrightarrow p \in A$ and $\varnothing \in \mathrm{C}^A(p) \leftrightarrow p\downarrow \cap A = \varnothing$.

Next, we have the *dense cover scheme* **Den** given by:

$$(*) \qquad S \in \mathbf{Den}(p) \leftrightarrow \forall q \le p \exists s \in S \exists r \le s(r \le q):$$

it is a straightforward exercise to show that this is a pretopology. When S is a sieve, the above condition (*) is easily seen to be equivalent to the familiar condition of *density below* p: that is, $\forall q \le p \exists s \in S(s \le q)$.

Note that the following are equivalent for any cover scheme **C:** (a) **C** $\lhd$ **Den,** (b) $\varnothing \notin$ **C**(p) for all p. For since $\varnothing \notin$ **Den**(p), (a) clearly implies (b). Conversely, assume (b), and let $S \in$ **C**(p). Then for each $q \le p$ there is $T \in$ **C**(q) for which $\forall t \in T\, \exists s \in S\, (t \le s)$. Since (by (b)) $T \neq \varnothing$, we may choose $t_0 \in T$ and $s_0 \in S$ for which $t_0 \le s_0$. Since $t_0 \le q$, and $q \le p$ was arbitrary, it follows that S satisfies the condition (*) above for membership in **Den**(p). This gives (a).

Finally, we have the *Beth cover scheme* **Bet**. This is defined as follows. First we define a *road* from p to be a maximal linearly preordered subset of $p{\downarrow}$: clearly any road from p contains p. Let us call a *rome* over p any subset of $p{\downarrow}$ intersecting every road from p. Now the Beth coverage has **Bet**(p) = collection of all romes over p. Let us check first that **Bet** is a cover scheme. Suppose that S is a rome over p and $q \le p$. We claim that

$$T = \{t \le q : \exists s \in S(t \le s)\}$$

is a rome over q. For let Y be any road from q; then, by Zorn's lemma, Y may be extended to a road X from p. We note that since $X \cap q{\downarrow}$ is linearly preordered and includes Y, it must coincide with Y. Since S is a rome over p, there must be an element $s \in S \cap X$. Since also $q \in Y \subseteq X$, we have $s \le q$ or $q \le s$. If $s \le q$, then $s \in X \cap q{\downarrow} = Y$ and $s \in T$, so that $s \in Y \cap T$. If $q \le s$, then $q \in T$; since $q \in Y$, it follows that $q \in Y \cap T$. So in either case $Y \cap T \neq \varnothing$; therefore T is a rome over q. Since clearly also $T \prec S$, we have shown that **Bet** is a cover scheme.

To show that **Bet** is a pretopology, we observe first that, for any p, $\{p\}$ is a rome over p. Now suppose that we are given: a rome S over p, for each $s \in S$, a rome T_s over s, and a road X from p. Then $s \in X \cap S$ for some s: we claim that $X \cap s{\downarrow}$ is a road from s. For suppose $t \le s$ is comparable with every member of $X \cap s{\downarrow}$; now since $s \in X$, for each $x \in X$ either $s \le x$ or $x \le s$. In the first case $t \le x$; in the second t is comparable with x by assumption. Hence t is comparable with every member of X, and so $t \in X$. Accordingly $X \cap s{\downarrow}$ is, as claimed, a road from s; as such, it must meet the rome T_s, so X meets $\bigcup_{s \in S} T_s$, and the latter is therefore a rome over p. So **Bet** is indeed a pretopology.

Since clearly $\varnothing \notin$ **Bet**(p) for any p, it follows from what we have noted above that **Bet** ⊲ **Den**, a fact that can also be easily verified directly.

Any preordered set (P, ≤) generates a *free lower semilattice* $\widetilde{P}$ which may be described as follows. The elements of $\widetilde{P}$ are the finite subsets of P; the preordering on $\widetilde{P}$ is the *refinement* relation $\sqsubseteq$, that is, for $F, G \in \widetilde{P}$,

$$F \sqsubseteq G \leftrightarrow \forall q \in G \exists p \in F (p \le q).$$

The meet operation $\wedge$ in $\widetilde{P}$ is set-theoretic union; the canonical embedding of P into $\widetilde{P}$ is the map $p \mapsto \{p\}$. Notice also that $\varnothing$ is the unique top element of $\widetilde{P}$.

Now, suppose we are given a cover scheme **C** on P. This induces a cover scheme $\tilde{\mathbf{C}}$ on $\widetilde{P}$ defined in the following way. We start by setting $\tilde{\mathbf{C}}(\varnothing) = \{\{\varnothing\}\}$. Now fix a nonempty finite subset F of P, take any nonempty subset $\{p_1,\ldots,p_n\}$ of F and any $S_1 \in \mathbf{C}(p_1),\ldots,S_n \in \mathbf{C}(p_n)$. Define

$$S_1 \bullet \ldots \bullet S_n = \{\{s_1,\ldots,s_n\} \cup F : s_1 \in S_1,\ldots,s_n \in S_n\}.$$

We decree that $\tilde{\mathbf{C}}(F)$ is to consist of all sets of the form $S_1 \bullet \ldots \bullet S_n$, for $S_1 \in \mathbf{C}(p_1),\ldots,S_n \in \mathbf{C}(p_n)$, and all nonempty finite subsets $\{p_1,\ldots,p_n\}$ of F.

Let us check that $\tilde{\mathbf{C}}$ is a cover scheme on $\widetilde{P}$. To begin with, the unique cover $\{\varnothing\}$ of $\varnothing$ is clearly sharpenable to any cover $S_1 \bullet \ldots \bullet S_n$ of any nonempty member of $\widetilde{P}$. Now suppose that $S_1 \bullet \ldots \bullet S_n$ is a $\tilde{\mathbf{C}}$-cover of a nonempty member F of $\widetilde{P}$ and that $G = \{q_1,\ldots,q_m\} \sqsubseteq F$. Then for each

$1 \leq i \leq n$ there is $q_i \in G$ for which $q_i \leq p_i$, hence $T_i \in \mathbf{C}(q_i)$ with $T_i \prec S_i$. Clearly $T_1 \bullet \ldots \bullet T_n \in \widetilde{\mathbf{C}}(G)$. Also $T_1 \bullet \ldots \bullet T_n \prec S_1 \bullet \ldots \bullet S_n$. For, given $t_1 \in T_1, \ldots, t_n \in T_n$, then since $T_i \prec S_i$ for each i, there are $s_1 \in S_1, \ldots, s_n \in S_n$ for which $t_1 \leq s_1, \ldots, t_n \leq s_n$, whence $\{t_1, \ldots, t_n\} \cup G \sqsubseteq \{s_1, \ldots, s_n\} \cup F$. So $\tilde{\mathbf{C}}$ satisfies the conditions of a cover scheme.

The normalization $\overline{\tilde{\mathbf{C}}}$ of $\tilde{\mathbf{C}}$ is then a coverage on $\widetilde{P}$ called the coverage on $\widetilde{P}$ *induced* by **C.**

We next show how cover schemes give rise to complete Heyting algebras, or frames (see, e.g. [3]).

A *Heyting algebra* is a lattice L with top and bottom elements 1, 0 such that, for any elements $x, y \in L$, there is an element—denoted by $x \Rightarrow y$—of L such that, for any $z \in L$,

$$z \leq x \Rightarrow y \text{ iff } z \wedge x \leq y.$$

Thus $x \Rightarrow y$ is the *largest* element z such that $z \wedge x \leq y$. So in particular, if we write $\neg x$ for $x \Rightarrow 0$, then $\neg x$ is the largest element z such that $x \Rightarrow z = 0$: it is called the *pseudocomplement* of x. A *Boolean algebra* is a Heyting algebra in which $\neg\neg x = x$ for all x, or equivalently, in which $x \vee \neg x = 1$ for all x.

If we think of the elements of a (complete) Heyting algebra as “truth values”, then 0, 1, $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\bigvee$, $\bigwedge$ represent “true”, “false”, “and”, “or”, “not” and “implies”, “there exists” and “for all”, respectively. The laws satisfied by these operations in a general Heyting algebra correspond to those of *intuitionistic logic.* In Boolean algebras the counterpart of the law of excluded middle also holds.

A basic fact about *complete* Heyting algebras is that the following identity holds in them:

(*) $$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I}(x \wedge y_i)$$

And conversely, in any complete lattice satisfying (*), defining the operation $\Rightarrow$ by $x \Rightarrow y = \bigvee\{z\colon z \wedge x \leq y\}$ turns it into a Heyting algebra.

In view of this result a complete Heyting algebra is frequently defined to be a complete lattice satisfying (*). A complete Heyting algebra is briefly called a *frame*.

Now we associate a frame with each cover scheme on *P*. First, we define $\widehat{P}$ to be the set of sieves in *P* partially ordered by inclusion: $\widehat{P}$ is then a frame—the *completion*[1] of *P*— in which joins and meets are just set-theoretic unions and intersections, and in which the operations $\Rightarrow$ and $\neg$ are given by

$$I \Rightarrow J = \{p : I \cap p\!\downarrow\subseteq J\} \qquad\qquad \neg I = \{p : I \cap p\!\downarrow = \varnothing\}.$$

Given a cover scheme **C** on *P*, a sieve *I* in *P* is said to be **C**-*closed* if

$$\exists S \in \mathbf{C}(p)(S \subseteq I) \rightarrow p \in I.$$

We write $\widehat{\mathbf{C}}$ for the set of all **C**-closed sieves in *P*, partially ordered by inclusion.

Lemma. If $I \in \widehat{P}$, $J \in \widehat{\mathbf{C}}$, then $I \Rightarrow J \in \widehat{\mathbf{C}}$.

Proof. Suppose that $I \in \widehat{P}$, $J \in \widehat{\mathbf{C}}$, and $S \subseteq I \Rightarrow J$ with $S \in \mathbf{C}(p)$. Define $U = \{q \in I\colon \exists s \in S.\ q \leq s\}$. Then $U \subseteq J$. If $q \in I \cap p\!\downarrow$, then there is $T \in \mathbf{C}(q)$ for which $T \prec S$. Then for any $t \in T$, there is $s \in S$ for which $t \leq s$, whence $t \in U$. Accordingly $T \subseteq U \subseteq J$. Since *J* is a **C**-closed, it follows that $q \in J$. We conclude that $I \cap p\!\downarrow \subseteq J$, whence $p \in p\!\downarrow \subseteq I \Rightarrow J$. Therefore $I \Rightarrow J$ is **C**-closed. □

[1] Writing **Lat** for the category of complete lattices and join preserving homomorphisms, $\widehat{P}$ is in fact the object in **Lat** freely generated by *P*.

It follows from the lemma that $\widehat{\mathbf{C}}$ is a frame. For clearly an arbitrary intersection of **C**-closed sieves is **C**-closed. So $\widehat{\mathbf{C}}$ is a complete lattice. In view of the lemma the implication operation in $\widehat{P}$ restricts to one in $\widehat{\mathbf{C}}$, making $\widehat{\mathbf{C}}$ a Heyting algebra, and so a frame.

Proposition 1. Suppose that **C** is a pretopology. Then **(i)** the bottom element of $\widehat{\mathbf{C}}$ is $\mathbf{0} = \{p: \varnothing \in \mathbf{C}(p)\}$, **(ii)** the **C**-closed sieve generated by a sieve A (that is, the smallest **C**-closed sieve containing A) is $\{p: \exists S \in \mathbf{C}(p).\ S \subseteq A\}$, **(iii)** the join operation in $\widehat{\mathbf{C}}$ is given by $\bigvee_{i \in I} J_i = (\bigcup_{i \in I} J_i)^*$. If **C** is a Grothendieck topology, then (**iv**) for any sieve $S \subseteq p{\downarrow}$, $p \in S^* \leftrightarrow S \in \mathbf{C}(p)$.

Proof. Suppose that **C** is a pretopology. Then **0** is a **C-**closed sieve. For it is easily seen to be a sieve; and it is **C-**closed because if $S \in \mathbf{C}(p)$ and $S \subseteq \mathbf{0}$, then $\varnothing \in \mathbf{C}(s)$ for each $s \in S$, whence $\varnothing = \bigcup_{s \in S} \varnothing \in \mathbf{C}(p)$, and so $p \in \mathbf{0}$. Finally, $\mathbf{0} \subseteq I$ for any **C**-closed sieve I, for if $\varnothing \in \mathbf{C}(p)$, then from $\varnothing \subseteq I$ we infer $p \in I$. This gives **(i).** As for **(ii)**, suppose given a sieve A. Then $A \subseteq A^*$ follows from $\{p\} \in \mathbf{C}(p)$. A^* is a sieve, since if $p \in A^*$ and $q \leq p$, then there is $S \in \mathbf{C}(p)$ for which $S \subseteq A$, and $T \in \mathbf{C}(q)$ sharpening S; clearly $T \subseteq A$ also, whence $q \in A^*$. And A^* is **C-**closed, since if $S \subseteq A^*$ with $S \in \mathbf{C}(p)$, then for each $s \in S$ there is $T_s \in \mathbf{C}(s)$ with $T_s \subseteq A$; it follows that $\bigcup_{s \in S} T_s \subseteq A$ and $\bigcup_{s \in S} T_s \in \mathbf{C}(p)$, whence $p \in A^*$. Part **(iii)** is an immediate consequence of **(iii).** Finally, if **C** is a Grothendieck topology and $S \subseteq p{\downarrow}$ is a sieve, then $p \in S^* \leftrightarrow \exists T \in \mathbf{C}(p)\ .\ T \subseteq S \leftrightarrow S \in \mathbf{C}(p)$, i.e. **(iv).** □

We observe parenthetically that $\widehat{\mathbf{Den}}$ *is a Boolean algebra.* To establish this it suffices to show that, for any $I \in \widehat{\mathbf{Den}}$, $\neg\neg I \subseteq I$. Now since $\varnothing \notin \mathbf{Den}(p)$, it follows from **(i)** of the proposition above that the bottom element of $\widehat{\mathbf{Den}}$ is $\varnothing$, so that, for any $I \in \widehat{\mathbf{Den}}$, $\neg I = \{p: I \cap p{\downarrow} = \varnothing\}$, whence $\neg\neg I = \{p : \forall q \le p \exists r \le q.\ r \in I\}$. But it easily checked that the defining condition for I to be a member of $\widehat{\mathbf{Den}}$ is precisely that, if $\forall q \le p \exists r \le q.\ r \in I$, then $p \in I$. That is, $\neg\neg I \subseteq I$.

Cover schemes on P correspond to certain self-maps on $\widehat{P}$ called (*weak*) *nuclei.* A *weak nucleus* on a frame H is a finite-meet-preserving map j: $H \to H$ such that $j(1) = 1$ and $a \le j(a)$ for any $a \in H$. If in addition $j(j(a)) \le j(a)$ (so that $j(j(a)) = j(a)$) for all $a \in H$, j is called a *nucleus* on H.

Proposition 2. Let $\mathbf{C}$ be a cover scheme on P. For each $I \in \widehat{P}$ let I^* be the least $\mathbf{C}$-closed sieve containing I. Then the map $k_{\mathbf{C}}$: $I \mapsto I^*$ is a nucleus on $\widehat{P}$.

Proof. Clearly $I \subseteq I^*$ and $I^{**} = I^*$. It remains to be shown that, for $I, J \in \widehat{P}$, $(I \cap J)^* = I^* \cap J^*$. Since * is obviously inclusion-preserving, $(I \cap J)^* \subseteq I^* \cap J^*$. For the reverse inclusion, note first that $I \in \widehat{\mathbf{C}} \leftrightarrow I^* = I$. Given $I, J \in \widehat{P}$, define $K = I \Rightarrow (I \cap J)^*$. By the Lemma above, $K \in \widehat{\mathbf{C}}$, so that $K^* = K$. Now $J^* \subseteq K$ since

$$J \cap I \subseteq (I \cap J)^* \to J \subseteq [I \Rightarrow (I \cap J)^*] = K,$$

whence $J^* \subseteq K^* \subseteq K$. Similarly, if we define $L = K \Rightarrow (I \cap J)^*$, then $I^* \subseteq L$. It follows that

$$I^* \cap J^* \subseteq K \cap L = K \cap [K \Rightarrow (I \cap J)^*] \subseteq (I \cap J)^*. \qquad \square$$

Inversely, any *weak* nucleus j on $\widehat{P}$ determines a *regular* cover scheme $\mathbf{D}_j$ on P, given by

$$S \in \mathbf{D}_j(p) \leftrightarrow p \in j(S).$$

Let us check that $\mathbf{D}_j$ is indeed a regular cover scheme. To do this it suffices to show that each $\mathbf{D}_j(p)$ is a filter of sieves and that, if $S \in \mathbf{D}_j(p)$, and $q \le p$, then $S \cap q{\downarrow} \in \mathbf{D}_j(q)$. The first of these properties follows immediately from the fact that j preserves finite intersections, and the second from the observation that, if $S \in \mathbf{D}_{j}(p)$, and $q \le p$, then $p \in j(S)$, so that $q \in j(S)$, and $q \in q{\downarrow} \subseteq j(q{\downarrow})$, whence $q \in j(S) \cap j(q{\downarrow}) = j(S \cap q{\downarrow})$, i.e. $S \cap q{\downarrow} \in \mathbf{D}_{j}(q)$.

When j is a *nucleus*, $\mathbf{D}_{j}$ is a *Grothendieck topology*. For under this assumption, if $S \in \mathbf{D}_{j}(p)$ and $T_s \in \mathbf{D}_{j}(s)$ for each $s \in S$, then $s \in j(T_s)$ for each $s \in S$, and it follows that

$$S \subseteq \bigcup_{s \in S} j(T_s) \subseteq j(\bigcup_{s \in S} T_s)$$

so that

$$p \in j(S) \subseteq j(j(\bigcup_{s \in S} T_s)) = j(\bigcup_{s \in S} T_s)$$

i.e., $\bigcup_{s \in S} T_s \in \mathbf{D}_{j}(p)$.

The correspondences $\mathbf{C} \mapsto k_{\mathbf{C}}$ and $j \mapsto \mathbf{D}_j$ between Grothendieck topologies on P and nuclei on $\widehat{P}$ are mutually inverse. For if $\mathbf{C}$ is a Grothendieck topology on P, then, by Proposition I **(iv)** we have

$$S \in \mathbf{D}_{k_{\mathbf{C}}}(p) \leftrightarrow p \in k_{\mathbf{C}}(S) = S^* \leftrightarrow S \in \mathbf{C}(p),$$

whence $\mathbf{D}_{k_{\mathbf{C}}} = \mathbf{C}$. And, for a nucleus j on $\widehat{P}$, we have, using Proposition 1**(ii)**,

$$\begin{aligned} k_{\mathbf{D}_j}(I) &= \text{ least } \mathbf{D}_j - \text{closed sieve } \supseteq I \\ &= \{p : \exists S \in \mathbf{D}_j(p). S \subseteq I\} \\ &= \{p : \exists S \subseteq I. p \in j(S)\} \\ &= j(I), \end{aligned}$$

whence $k_{\mathbf{D}_j} = j$.

II. COVER SCHEMES AND FRAMES

The relationship between cover schemes on a preordered set and (weak) nuclei on its completion can be extended to cover schemes on partially ordered sets and general frames. Accordingly let H be a frame: we write $\bigvee$, $\bigwedge$, $\Rightarrow$ for the join, meet and implication operations, respectively, in H. The partially ordered set $(P, \leq)$ is said to be *dense* in H if P is a subset of H, the partial ordering on P is the restriction to P of that of H, and either of the two following equivalent conditions is satisfied: (i) for any $a \in H$, $a = \bigvee\{p: a \leq p\}$ (ii) for any $a, b \in H$, $a \leq b \leftrightarrow \forall p[p \leq a \rightarrow p \leq b]$. The canonical example of a frame in which P is dense is the frame $\widehat{P}$ described in section I: here each $p \in P$ is identified with the $p\!\downarrow\, \in \widehat{P}$. $\widehat{P}$ is easily seen to have the property that in it, for any $S \subseteq P$, $p \leq \bigvee S$ iff $p \in S$.

Now fix a frame H in which P is dense and a cover scheme $\mathbf{C}$ on P. An element $a \in H$ is said to *cover* an element $p \in P$ if there exists a cover S of p for which $\bigvee S \leq a$. A $\mathbf{C}$-*element* of H is one which dominates every element of P that it covers—that is, an element $a \in H$ satisfying

$$\forall p \in P[(\exists S \in \mathbf{C}(p))\bigvee S \leq a \rightarrow p \leq a].$$

We write $H_{\mathbf{C}}$ for the set of all $\mathbf{C}$-elements of H. It is evident that $H_{\mathbf{C}}$ is closed under the meet operation of H. Notice that $\mathbf{C}$-elements and $\overline{\mathbf{C}}$-elements coincide (recalling that $\overline{\mathbf{C}}$ is the normalization of $\mathbf{C}$.)

The *canonical H-cover scheme* $\mathbf{C}_H$ on P is given by

$$S \in \mathbf{C}_H(p) \leftrightarrow \bigvee S = p.$$

Clearly $\mathbf{C}_H$ is a pretopology, and every element of H is a $\mathbf{C}_H$-element.

Corresponding to the Lemma of §I, we have:

Lemma. If $a \in H, b \in H_{\mathbf{C}}$, then $a \Rightarrow b \in H_{\mathbf{C}}$.

Proof. Suppose $a \in H, b \in H_{\mathbf{C}}$, $S \in C(p)$ and $\bigvee S \leq (a \Rightarrow b)$. Writing U for $\{q : q \leq a \;\&\; \exists s \in S(q \leq s)\}$, we have

$$\bigvee U \leq \bigvee\{s \wedge q : s \in S, q \leq a\} = \bigvee S \wedge \bigvee\{q : q \leq a\} = \bigvee S \wedge a \leq b.$$

Now if $q \leq p \wedge a$, there is $T \in \mathbf{C}(q)$ sharpening S. Then

$$t \in T \rightarrow t \leq a \;\&\; \exists s \in S(t \leq s),$$

so that $T \subseteq U$, and therefore $\bigvee T \leq \bigvee U \leq b$. Since $b \in H_{\mathbf{C}}$, it follows that $q \leq b$. Hence $q \leq p \wedge a \rightarrow q \leq b$, so that $p \wedge a \leq b$ and $p \leq (a \Rightarrow b)$. We conclude that $(a \Rightarrow b) \in H_{\mathbf{C}}$. □

It follows from the lemma that $H_{\mathbf{C}}$ is itself a frame.

The nucleus on H *associated* with the cover scheme $\mathbf{C}$ on P is the map $j = k_{\mathbf{C}}$: $H \rightarrow H$ defined by

$$j(a) = \bigwedge\{x \in H_{\mathbf{C}} : a \leq x\}.$$

That j is a nucleus results from the following observations. Evidently j is order preserving, maps H onto $H_{\mathbf{C}}$, is the identity on $H_{\mathbf{C}}$, and satisfies $j(1) = 1$ and $a \leq j(a)$ for all $a \in A$. Also it is easily shown that $j(j(a)) = j(a)$. Finally, j preserves finite meets. For clearly $j(a \wedge b) \leq j(a) \wedge j(b)$ since j is order preserving. For the reverse inequality, consider first the element $u = (a \Rightarrow j(a \wedge b))$: this is, by the Lemma above, an element of $H_{\mathbf{C}}$, so that $j(u) = u$. Also $j(b) \leq u$. For from $b \wedge a \leq j(a \wedge b)$ we deduce $b \leq (a \Rightarrow j(a \wedge b)) = u$, whence $j(b) \leq j(u) = u$. Similarly, $v = ((a \Rightarrow j(a \wedge b)) \Rightarrow j(a \wedge b))$ is an element of H_C and $j(a) \leq v$. Therefore

$$j(a) \wedge j(b) \leq v \wedge u \leq j(a \wedge b),$$

as required.

Notice that the nucleus associated with a cover scheme coincides with that associated with its normalization.

Accordingly we have shown that each cover scheme on P determines a nucleus on H. Conversely, we can show that any *weak nucleus* on H determines a cover scheme on P. For, starting with a weak nucleus j on H, define the map $\mathbf{D}_j$ on P by

$$\mathbf{D}_j(p) = \{S \subseteq p\downarrow : p \leq j(\bigvee S)\}.$$

Then $\mathbf{D}_j$ is a cover scheme on P. For suppose $q \leq p$ and $S \in \mathbf{D}_j(p)$. Then $q \leq p \leq j(\bigvee S)$; since $q \leq j(q)$ and j preserves finite meets, it follows that

(*) $$q \leq j(q) \wedge j(\bigvee S) = j(q \wedge \bigvee S) = j(\bigvee\{s \wedge q : s \in S\}).$$

Now define $T \subseteq q{\downarrow}$ by

$$T = \{t : t \le q \;\&\; \exists s \in S(t \le s)\}.$$

We claim that T is a ($\mathbf{D}_j$-) cover of q sharpening S. That T sharpens S is evident from its definition. To see that it is a cover of q we observe that, if $s \in S$, then

$$s \wedge q = \bigvee\{t : t \le s \wedge q\} = \bigvee\{t : t \le s \;\&\; t \le q\} \le \bigvee T.$$

Therefore $\bigvee\{s \wedge q : s \in S\} \le \bigvee T$, so that, by (*),

$$q \le j(\bigvee\{s \wedge q : s \in S\}) \le j(\bigvee T),$$

that is, T covers q.

When j is a *nucleus,* the associated cover scheme $\mathbf{D}_j$ is actually a *pretopology.* For in any case $\{p\} \in \mathbf{D}_j(p)$. Moreover, if j is a nucleus, $S \in \mathbf{D}_j(p)$ and $T_s \in \mathbf{D}_j(s)$ for each $s \in S$, then

$$p \le j(\bigvee S) \le j(\bigvee_{s\in S} j(\bigvee T_s)) \le j(j(\bigvee_{s\in S} \bigvee T_s) = j(\bigvee \bigcup_{s\in S} T_s).$$

Therefore $\bigcup_{s\in S} T_s \in \mathbf{D}_j(p)$, and $\mathbf{D}_j$ is a pretopology.

Starting with a weak nucleus j, we obtain the corresponding cover scheme $\mathbf{D}_j$. The latter in turn determines a nucleus j^* given by

$$j^*(a) = \bigwedge\{x \in H_{\mathbf{D}_j} : a \le x\}.$$

Now by definition, we have

$$\begin{aligned} x \in H_{\mathbf{D}_j} &\leftrightarrow \forall p[(\exists S \in \mathbf{D}_j(p)) \bigvee S \le x \to p \le x] \\ &\leftrightarrow \forall p[(\exists S \subseteq p{\downarrow})(p \le j(\bigvee S) \;\&\; \bigvee S \le x) \to p \le x] \qquad \text{(a)} \\ &\leftrightarrow \forall p[p \le j(x) \to p \le x] \qquad \text{(b)} \\ &\leftrightarrow j(x) = x. \end{aligned}$$

(To see the equivalence between (a) and (b), we need to establish the equivalence between $(\exists S \subseteq p{\downarrow})(p \le j(\bigvee S) \;\&\; \bigvee S \le x)$ and $p \le j(x)$. Clearly the first of these implies the second. As for the converse, if $p \le j(x)$, then since $p \le j(p)$, we have

$$p \le j(x) \wedge j(p) = j(x \wedge p) = j(\bigvee S),$$

where $S = \{q : q \le x \wedge p\}$. Then $S \subseteq p\downarrow$, $p \le j(\bigvee S)$ and $\bigvee S \le x$, and the first statement follows.) Accordingly

(*) $$j^*(a) = \bigwedge\{x \in H : a \le x \;\&\; j(x) = x\}.$$

j^* is called the nucleus[2] *generated* by the weak nucleus j; it is easily deduced from (*) that when j is a nucleus, j^* and j coincide.

The generation of nuclei by weak nuclei can itself be seen as an instance of a nuclear operation. For consider the set $W(H)$ of all weak nuclei on H. When $W(H)$ is partially ordered pointwise in the obvious way, it becomes a frame with implications, joins, and meets given by the following specifications: $(j \Rightarrow k)(a) = \bigwedge_{b \ge a}(j(b) \Rightarrow k(b))$ and for $S \subseteq W(H)$, $(\bigvee S)(a) = \bigvee_{s \in S} s(a)$, $(\bigwedge S)(a) = \bigwedge_{s \in S} s(a)$. The subset $N(H)$ of $W(H)$ consisting of all nuclei can be shown to be a sublocale (see [3]) of $N(H)$, that is, it is closed under arbitrary meets in $W(H)$ and is such that $(j \Rightarrow k) \in N(H)$ whenever $j \in W(H)$, $k \in N(H)$. That being the case, the map φ: $W(H) \to N(H)$ defined by

$$\varphi(j) = \bigwedge\{k \in N(H) : j \le k\}$$

is a nucleus on $W(H)$, and it is easily shown that $\varphi(j) = j^*$. So the generation of nuclei by weak nuclei is precisely the action of the nucleus φ.

Now start with a cover scheme $\mathbf{C}$ on P, obtain the associated nucleus $k_{\mathbf{C}}$ on H, and consider its associated cover scheme $\mathbf{D}_{k_{\mathbf{C}}} = \mathbf{C}^*$ on P. By definition we have, for $S \subseteq p\downarrow$,

$$\begin{aligned} S \in \mathbf{C}^*(p) &\leftrightarrow p \le j_{\mathbf{C}}(\bigvee S) \\ &\leftrightarrow p \le \bigwedge\{x \in H_{\mathbf{C}} : \bigvee S \le x\} \\ &\leftrightarrow \forall x \in H_{\mathbf{C}}(\bigvee S \le x \to p \le x) \end{aligned}$$

Recalling the definition of $H_{\mathbf{C}}$, we see immediately that this last assertion is implied by $S \in \mathbf{C}(p)$, so that always $\mathbf{C}(p) \subseteq \mathbf{C}^*(p)$. The reverse inclusion

[2] It can be verified directly that j^* is a nucleus.

will hold, and so **C** will coincide with **C***, precisely when the cover scheme **C** is *saturated*, that is, coincides with its *saturate*, which we next proceed to define.

The (H-) *saturate* $\widetilde{\mathbf{C}}$ of a cover scheme C on P is defined by

$$\widetilde{\mathbf{C}}(p) = \{S \subseteq p\downarrow : \forall x \in H_{\mathbf{C}}(\bigvee S \leq x \rightarrow p \leq x)\}.$$

Then $\widetilde{\mathbf{C}}$ is a cover scheme. For if $S \in \widetilde{\mathbf{C}}(p)$ and $q \leq p$, consider the subset T of $p\downarrow$ defined by

$$T = \{t \leq q : \exists s \in S(t \leq s)\}.$$

It is easily shown that $\bigvee T = (\bigvee S) \wedge q$. Now if $x \in H_{\mathbf{C}}$ and $\bigvee T \leq x$, then $\bigvee S \wedge q \leq x$, whence $\bigvee S \leq (q \Rightarrow x)$. But since x is an element of $H_{\mathbf{C}}$, so, by the lemma, is $q \Rightarrow x$, and since $S \in \widetilde{\mathbf{C}}(p)$, it follows that $p \leq (q \Rightarrow x)$. Thus $q = p \wedge q \leq x$. Accordingly $T \in \widetilde{\mathbf{C}}(q)$, and T obviously sharpens S. This shows that $\widetilde{\mathbf{C}}$ is indeed a cover scheme.

It is readily shown that any cover scheme associated with a nucleus (as opposed to a weak nucleus) is saturated. Observe that, when H is $\widehat{P}$, every coverage on P is saturated, since in that case $H_{\mathbf{C}}$ is $\widehat{\mathbf{C}}$ and so we have, using Proposition I.1 (**iv**),

$$S \in \widetilde{\mathbf{C}}(p) \leftrightarrow \forall I \in \widehat{\mathbf{C}}[S \subseteq I \rightarrow p \in I] \leftrightarrow p \in S^* \leftrightarrow S \in \mathbf{C}(p).$$

To sum up, each weak nucleus on H gives rise to a cover scheme on P and the cover scheme associated with a nucleus is saturated. Conversely, each cover scheme gives rise to a nucleus. This establishes mutually inverse correspondences between nuclei and saturated cover schemes.

Given $a \in H$, we define the nuclei j_a j^a on H by

$$j_a(x) = a \vee x \qquad j^a(x) = a \Rightarrow x.$$

The associated cover schemes (easily seen to be pretopologies) on P are given by:

$$S \in \mathbf{C}_a(p) \leftrightarrow p \leq a \vee \bigvee S$$
$$S \in \mathbf{C}^a(p) \leftrightarrow p \wedge a \leq \bigvee S.$$

Notice that

$$p \leq a \leftrightarrow \varnothing \in \mathbf{C}_a(p)$$
$$p \leq \neg a \leftrightarrow \varnothing \in \mathbf{C}^a(p).$$

The *double negation operation* $\neg\neg$ is a nucleus on H, whose associated cover scheme is precisely the dense cover scheme **Den** (which accordingly is also known as the *double negation cover scheme*). An argument similar to the one above showing that $\widehat{\mathbf{Den}}$ is a Boolean algebra establishes that $H_{\mathbf{Den}}$ is a Boolean algebra: it is in fact the complete Boolean algebra of $\neg\neg$-closed elements of H.

Proposition. Let j be a weak nucleus on H. Then the following are equivalent: (a) $j(0) = 0$ (b) $j \leq \neg\neg$ (in the pointwise ordering of $W(H)$) (c) $\varnothing \notin \mathbf{D}_j(p)$ for all p.

Proof. If $j \leq \neg\neg$ then $j0 \leq \neg\neg 0 = 0$. Conversely if $j0 = 0$ then, for any $a \in H$, $j(a) \wedge \neg a \leq j(a) \wedge j(\neg a) = j(a \wedge \neg a) = j0 = 0$. So $j(a) \leq \neg\neg a$. Finally, we have

$$\begin{aligned} j0 = 0 \leftrightarrow 0 \in H_{\mathbf{D}_j} &\leftrightarrow \forall p[(\exists S \in \mathbf{D}_j(p))\bigvee S = 0 \to p = 0] \\ &\leftrightarrow \forall p[\neg(\exists S \in \mathbf{D}_j(p)\bigvee S = 0] \\ &\leftrightarrow \forall p[\varnothing \notin \mathbf{D}_j(p)]. \qquad \square \end{aligned}$$

III. COVER SCHEMES AND KRIPKE MODELS

We revert to the assumption that $(P, \leq)$ is a preordered set. Recall that a *presheaf* on P is an assignment, to each $p \in P$, of a set $\mathscr{F}(p)$ and to each pair (p, q) with $q \leq p$ of a map $\mathscr{F}_{pq}$: $\mathscr{F}(p) \to \mathscr{F}(q)$ in such a way that $\mathscr{F}_{pp}$ is the identity on $\mathscr{F}(p)$ and, for $r \leq q \leq p$, $\mathscr{F}_{pr} = \mathscr{F}_{qr} \circ \mathscr{F}_{pq}$. The set $V(\mathscr{F})$ =

$\bigcup_{p\in P}\mathscr{F}(p)$ is called the *universe* of $\mathscr{F}$. A *Kripke model* based on P is a presheaf $\mathscr{K}$ for which $\mathscr{K}(p) \subseteq \mathscr{K}(q)$ whenever $q \leq p$ and each $\mathscr{K}_{pq}$ is the corresponding insertion map. Put more simply, a Kripke model based on P is a map $\mathscr{K}$ from P to a family of sets satisfying $\mathscr{K}(p) \subseteq \mathscr{K}(q)$ whenever $q \leq p$. A Kripke model $\mathscr{K}$ based on P may be regarded as a set "evolving" or "growing" over P: each $\mathscr{K}(p)$ may be thought of as the "state" of the evolving set $\mathscr{K}$ at "stage" p.

Now suppose that we are given a cover scheme **C** on P. A Kripke model $\mathscr{K}$ based on P satisfying

$$\mathscr{K}(p) = \bigcap_{s\in S} \mathscr{K}(s)$$

for any $p \in P$, $S \in \mathbf{C}(p)$ is said to be *compatible with* **C**. (When P is directed downward, that is, whenever each pair of elements of P has a lower bound, and **C** is a pretopology on P, a Kripke model compatible with **C** is nothing other than a **C**-*sheaf*.)

Each Kripke model $\mathscr{K}$ based on P induces a Kripke model $\widetilde{\mathscr{K}}$ based on the free lower semilattice $\widetilde{P}$ generated by P by setting

$$\widetilde{\mathscr{K}}(\varnothing) = \varnothing \qquad \widetilde{\mathscr{K}}(\{p_1,\dots p_n\}) = \mathscr{K}(p_1) \cup \dots \cup \mathscr{K}(p_n).$$

If, further, $\mathscr{K}$ is compatible with the cover scheme **C** on P, then $\widetilde{\mathscr{K}}$ is compatible with the cover scheme on $\widetilde{P}$ induced by P (and hence also with the associated coverage on $\widetilde{P}$.) For suppose that $\mathscr{K}$ is in fact compatible with the cover scheme **C** on P. Given $F \in \widetilde{P}$, a nonempty subset $\{p_1,\dots, p_n\}$ of F, and $S_1 \in \mathbf{C}(p_1),\dots,S_n \in \mathbf{C}(p_n)$, we have

$$
\begin{aligned}
\bigcap_{X\in S_1\bullet\ldots\bullet S_n} \widetilde{\mathcal{K}}(X) &= \bigcap_{s_1\in S_1,\ldots,s_n\in S_n} \widetilde{\mathcal{K}}(\{s_1,\ldots,s_n\}\cup F)\\
&= \bigcap_{s_1\in S_1,\ldots,s_n\in S_n} (\mathcal{K}(s_1)\cup\ldots\cup\mathcal{K}(s_n)\cup\bigcup_{p\in F}\mathcal{K}(p))\\
&= \bigcap_{s_1\in S_1}\mathcal{K}(s_1)\cup\ldots\cup\bigcap_{s_n\in S_n}\mathcal{K}(s_n)\cup\bigcup_{p\in F}\mathcal{K}(p))\\
&= \mathcal{K}(p_1)\cup\ldots\cup\mathcal{K}(p_n)\cup\bigcup_{p\in F}\mathcal{K}(p)\\
&= \widetilde{\mathcal{K}}(F).
\end{aligned}
$$

Now suppose that the cover scheme **C** is in fact a *pretopology.* Then any Kripke model $\mathcal{K}$ based on P induces a Kripke model $\mathcal{K}_C$ also based on P but in addition compatible with **C** given by

$$\mathcal{K}_{\mathbf{C}}(p) = \bigcup_{S\in\mathbf{C}(p)}\bigcap_{s\in S}\mathcal{K}(s),$$

that is ,

$$a\in\mathcal{K}_{\mathbf{C}}(p)\leftrightarrow\exists S\in\mathbf{C}(p)\forall s\in S.\ a\in\mathcal{K}(s).$$

We note that $\mathcal{K}(p)\subseteq\mathcal{K}_{\mathbf{C}}(p)$ for every p. It is easily checked that this defines a Kripke model over P; let us confirm its compatibility with **C.** It suffices to show that, given $S\in\mathbf{C}(p)$, we have $\bigcap_{s\in S}\mathcal{K}_{\mathbf{C}}(s)\subseteq\mathcal{K}_{\mathbf{C}}(p)$. Indeed, if $a\in\bigcap_{s\in S}\mathcal{K}_{\mathbf{C}}(s)$,. then for each $s\in S$ there is $T_s\in\mathbf{C}(s)$ with $a\in\bigcap_{t\in T_s}\mathcal{K}(t)$. Writing $T=\bigcup_{s\in S}T_s$, we then have $T\in\mathbf{C}(p)$ and $a\in\bigcap_{t\in T}\mathcal{K}(t)$. It follows that $a\in\mathcal{K}_{\mathbf{C}}(p)$, as required.

Let us now examine some special cases. Let U be a subset of the universe V of $\mathcal{K}$, and let U^* be the sieve $\{p\colon U\subseteq\mathcal{K}(p)\}$. Now consider the Kripke model $\mathcal{K}^U$ compatible with $\mathbf{C}^{U^*}$ induced by $\mathcal{K}$. For arbitrary $p\in P$, we have $S=p{\downarrow}\cap U^*\in\mathbf{C}^{U^*}(p)$. and $U\subseteq\mathcal{K}(s)\subseteq\mathcal{K}^U(s)$ for every $s\in S$. Hence $U\subseteq\bigcap_{s\in S}\mathcal{K}^U(s)=\mathcal{K}^U(p)$. Thus, under these conditions, U is a subset of *every* $\mathscr{K}^U(p)$. In other words, the passage from $\mathscr{K}$ to $\mathscr{K}^U$ *forces* U to be included in the state of $\mathscr{K}^U$ at each stage. (Note that if $U^* = \varnothing$ then $\mathscr{K}^U$ assumes the constant value V.)

Again let U be a subset of V; this time define U^+ to be the sieve $\{p: U \cap \mathscr{K}(p) \neq \varnothing\}$. Now consider the Kripke model $\mathscr{K}_U$ compatible with $\mathbf{C}_{U+}$ induced by $\mathscr{K}$. Then for any p we have

$$U \cap \mathscr{K}(p) \neq \varnothing \to p \in U^+ \to \varnothing \in \mathbf{C}_{U+}(p) \to \mathscr{K}_U(p) = \bigcap_{s \in \varnothing} \mathscr{K}_U(s) = V.$$

That is, the passage from $\mathscr{K}$ to $\mathscr{K}_U$ *forces* each state of $\mathscr{K}_U$, apart from those already maximal, to be disjoint from U. (Notice that if $U^+ = P$, then $\mathscr{K}_U$ assumes the constant value V.)

We next turn to *logic* in Kripke models. Each Kripke model $\mathscr{K}$ based on P, with universe V, determines a map $\widehat{\mathscr{K}} : V \to \widehat{P}$ given by

$$\widehat{\mathscr{K}}(v) = \{p : v \in \mathscr{K}(p)\}.$$

This extends naturally to a homomorphism—also denoted by $\widehat{\mathscr{K}}$—of the free Heyting algebra $F(V)$ generated by V into $\widehat{P}$. Think of the members of $F(V)$ as the formulas of intuitionistic propositional logic generated by the members of V regarded as propositional atoms. Introduce the familiar *forcing* relation $\Vdash_{\mathscr{K}}$ between P and $F(V)$ by defining

$$(*) \qquad p \Vdash_{\mathscr{K}} \varphi \leftrightarrow p \in \widehat{\mathscr{K}}(\varphi).$$

Then the fact that $\widehat{\mathscr{K}} : F(V) \to \widehat{P}$ is a homomorphism of Heyting algebras translates into the usual rules for "Kripke semantics", namely

- $p \Vdash_{\mathscr{K}} \varphi \wedge \psi \leftrightarrow p \Vdash_{\mathscr{K}} \varphi$ & $p \Vdash_{\mathscr{K}} \psi$
- $p \Vdash_{\mathscr{K}} \varphi \vee \psi \leftrightarrow p \Vdash_{\mathscr{K}} \varphi$ or $p \Vdash_{\mathscr{K}} \psi$
- $p \Vdash_{\mathscr{K}} \varphi \Rightarrow \psi \leftrightarrow \forall q \leq p[\, q \Vdash_{\mathscr{K}} \varphi \to q \Vdash_{\mathscr{K}} \psi]$
- $p \Vdash_{\mathscr{K}} \neg\varphi \leftrightarrow \forall q \leq p \;\, q \nVdash_{\mathscr{K}} \varphi$

Equally, the map $\widehat{\mathscr{K}} : V \to \widehat{P}$ extends to a frame homomorphism (i.e., a map preserving top elements, $\wedge$, and $\bigvee$)—again denoted by $\widehat{\mathscr{K}}$—of the *free frame* $\Phi(V)$ generated by V. Think of the members of $\Phi(V)$ as the formulas of *infinitary* intuitionistic propositional logic generated by the members of V regarded as propositional atoms. Such a formula φ is said to be *geometric* if it is generated from propositional atoms by applying just $\wedge$ and $\bigvee$. Introducing the forcing relation $\Vdash_{\mathscr{K}}$ between P and $\Phi(V)$ as in (*) above, the fact that $\widehat{\mathscr{K}} : \Phi(V) \to \widehat{P}$ is a frame homomorphism translates into the semantical rules for *geometric* formulas:

- $p \Vdash_{\mathscr{K}} \varphi \wedge \psi \leftrightarrow p \Vdash_{\mathscr{K}} \varphi \ \ \& \ \ p \Vdash_{\mathscr{K}} \psi$
- $p \Vdash_{\mathscr{K}} \bigvee_{i \in I} \varphi_i \leftrightarrow \exists i \in I \ \ p \Vdash_{\mathscr{K}} \varphi_i$.

Now suppose that $\mathbf{C}$ is a pretopology on P. It is then easily seen that $\mathscr{K}$ is compatible with $\mathbf{C}$ iff each $\widehat{\mathscr{K}}(v)$ is a $\mathbf{C}$-closed sieve. So if $\mathscr{K}$ is compatible with $\mathbf{C}$, the resulting map $\widehat{\mathscr{K}} : V \to \widehat{\mathbf{C}}$ can be extended to a homomorphism, which we shall denote by $\widehat{\mathscr{K}}_{\mathbf{C}}$, of $F(V)$ into $\widehat{\mathbf{C}}$. Introducing the forcing relation $\Vdash_{\mathscr{K},\mathbf{C}}$ between P and $F(V)$ by

(**) $$p \Vdash_{\mathscr{K},\mathbf{C}} \varphi \leftrightarrow p \in \widehat{\mathscr{K}_{\mathbf{C}}}(\varphi),$$

we find that the fact that $\widehat{\mathscr{K}}_{\mathbf{C}} : F(V) \to \widehat{\mathbf{C}}$ is a homomorphism translates into the rules of "Beth-Kripke-Joyal" semantics for $\Vdash_{\mathscr{K},\mathbf{C}}$ (see, e.g., [4]), viz.,

- $p \Vdash_{\mathscr{K},\mathbf{C}} \varphi \wedge \psi \leftrightarrow p \Vdash_{\mathscr{K},\mathbf{C}} \varphi \ \ \& \ \ p \Vdash_{\mathscr{K},\mathbf{C}} \psi$
- $p \Vdash_{\mathscr{K},\mathbf{C}} \varphi \vee \psi \leftrightarrow \exists S \in \mathbf{C}(p) \ \forall s \in S \ \ [s \Vdash_{\mathscr{K},\mathbf{C}} \varphi \ \text{ or } \ s \Vdash_{\mathscr{K},\mathbf{C}} \psi]$
- $p \Vdash_{\mathscr{K},\mathbf{C}} \varphi \Rightarrow \psi \ \leftrightarrow \ \forall q \leq p [\ q \Vdash_{\mathscr{K},\mathbf{C}} \varphi \ \to \ q \Vdash_{\mathscr{K},\mathbf{C}} \psi]$
- $p \Vdash_{\mathscr{K},\mathbf{C}} \neg\varphi \ \leftrightarrow \ \forall q \leq p \ \ [q \Vdash_{\mathscr{K},\mathbf{C}} \varphi \to \varnothing \in \mathbf{C}(p)]$.

We verify the second and fourth of these. We have, using Proposition I 1. (iii),

$$p \Vdash_{\mathscr{K},\mathbf{C}} \varphi \vee \psi \leftrightarrow p \in \widehat{\mathscr{K}_{\mathbf{C}}}(\varphi \vee \psi) = \widehat{\mathscr{K}_{\mathbf{C}}}(\varphi) \vee \widehat{\mathscr{K}_{\mathbf{C}}}(\psi) \text{ (in } \widehat{\mathbf{C}}\text{)}$$

$$\leftrightarrow \exists S \in \mathrm{C}(p).\ S \subseteq \widehat{\mathscr{K}_{\mathbf{C}}}(\varphi) \cup \widehat{\mathscr{K}_{\mathbf{C}}}(\psi)$$

$$\leftrightarrow \exists S \in \mathrm{C}(p) \forall s \in S.\ s \in \widehat{\mathscr{K}_{\mathbf{C}}}(\varphi) \vee s \in \widehat{\mathscr{K}_{\mathbf{C}}}(\psi)$$

$$\leftrightarrow \exists S \in \mathbf{C}(p)\ \forall s \in S\ [s \Vdash_{\mathscr{K},\mathbf{C}} \varphi \text{ or } s \Vdash_{\mathscr{K},\mathbf{C}} \psi].$$

Also, using Proposition I 1. (ii) we have

$$p \Vdash_{\mathscr{K},\mathbf{C}} \neg\varphi \leftrightarrow p \in \widehat{\mathscr{K}_{\mathbf{C}}}(\neg\varphi) = \neg\widehat{\mathscr{K}_{\mathbf{C}}}(\varphi) = (\widehat{\mathscr{K}_{\mathbf{C}}}(\varphi) \Rightarrow \mathbf{0})$$

$$\leftrightarrow \forall q \le p[q \in \widehat{\mathscr{K}}_{\mathbf{C}}(\varphi) \rightarrow \varnothing \in \mathbf{C}(q)]$$

$$\leftrightarrow \forall q \le p[q \Vdash_{\mathscr{K},\mathbf{C}} \varphi \rightarrow \varnothing \in \mathbf{C}(q)].$$

Since $\widehat{\mathbf{Den}}$ is a Boolean algebra it follows that, when $\mathscr{K}$ is compatible with **Den**, $p \Vdash_{\mathscr{K},\mathbf{Den}} \varphi \vee \neg\varphi$ for every p, i.e., classical logic prevails in the Kripke model associated with $\widehat{\mathscr{K}}_{\mathbf{Den}}$.

When $\mathscr{K}$ is compatible with **C**, the map $\widehat{\mathscr{K}} : V \rightarrow \widehat{\mathbf{C}}$ can be extended to a frame homomorphism, which we shall again denote by $\widehat{\mathscr{K}}_{\mathbf{C}}$, of $\Phi(V)$ into $\widehat{\mathbf{C}}$. Introduce the forcing relation $\Vdash_{\mathscr{K},\mathbf{C}}$, now between P and $\Phi(V)$, by the same equivalence (**) as above. When **C** is a Grothendieck topology, a straightforward inductive argument shows that, for any geometric formula φ,

(†) $$p \Vdash_{\mathscr{K},\mathbf{C}} \varphi \leftrightarrow \exists S \in \mathbf{C}(p)\ \forall s \in S.\ s \Vdash_{\mathscr{K}} \varphi.$$

This may be applied to "force" any given set Σ of geometric formulas to become true in a Kripke model. For, starting with a Kripke model $\mathscr{K}$, let A be the sieve $\{p\colon \forall\sigma\in\Sigma.\ p \Vdash_{\mathscr{K}} \sigma\}$. Let **G** be the Grothendieck topology generated by the coverage $\mathbf{C}^A$: it is easily verified that a sieve $S \subseteq p\downarrow$

satisfies the same condition for membership in $\mathbf{G}(p)$ as in $\mathbf{C}^A(p)$, viz., $p\!\downarrow \cap A \subseteq S$. Now by (†) we have, for each $\sigma \in \Sigma$,

$$(\ddagger) \qquad p \Vdash_{\mathscr{K},\mathbf{G}} \sigma \;\leftrightarrow\; \exists S \in \mathbf{G}(p)\ \forall s \in S\ .\ s \Vdash_{\mathscr{K}} \sigma.$$

If we take S to be $p\!\downarrow \cap A$, then evidently $S \in \mathbf{G}(p)$ and $\forall s \in S\ .\ s \Vdash_{\mathscr{K}} \sigma$. It now follows from (‡) that $p \Vdash_{\mathscr{K},\mathbf{G}} \sigma$ for every $\sigma \in \Sigma$ and every $p \in P$. In this sense $\mathbf{G}$ "forces" all the members of Σ to be true in the Kripke model associated with $\widehat{\mathscr{K}_{\mathbf{G}}}$.

IV. COVER SCHEMES AND FRAME-VALUED SET THEORY

We now set about relating what has been done so far to frame-valued set theory. Associated with each frame H is an H-valued model $V^{(H)}$ of (intuitionistic) set theory (see, e.g. [1] or [2]): we recall some of its principal features.

- Each of the members of $V^{(H)}$—the *H-valued sets*—is a map from a subset of $V^{(H)}$ to H.
- Corresponding to each sentence σ of the language of set theory (with names for all elements of $V^{(H)}$) is an element $[\![\sigma]\!] = [\![\sigma]\!]^H \in H$ called its *truth value in* $V^{(H)}$. These "truth values" satisfy the following conditions. For $a, b \in V^{(H)}$,

$$[\![b \in a]\!] = \bigvee_{c \in dom(a)} [\![b = c]\!] \wedge a(c)$$

$$[\![b = a]\!] = \bigvee_{c \in dom(a) \cup dom(b)} ([\![c \in b]\!] \Leftrightarrow [\![c \in a]\!])$$

$$[\![\sigma \wedge \tau]\!] = [\![\sigma]\!] \wedge [\![\tau]\!], \text{ etc.}$$

$$[\![\exists x \varphi(x)]\!] = \bigvee_{a \in V^{(H)}} [\![\varphi(a)]\!]$$

$$[\![\forall x \varphi(x)]\!] = \bigwedge_{a \in V^{(H)}} [\![\varphi(a)]\!]$$

A sentence σ is *valid* , or *holds,* in $V^{(H)}$, written $V^{(H)} \vDash \sigma$, if $[\![\sigma]\!] = 1$, the top element of H. The truth value $[\![\sigma]\!]$ "measures" the degree or extent to which σ holds: the larger $[\![\sigma]\!]$ is, the "truer" σ is. In particular, when $[\![\sigma]\!] = 1$, σ is 'universally" or "absolutely" true, and when $[\![\sigma]\!] = 0$, σ is "universally" or "absolutely" false.

- The axioms of intuitionistic Zermelo-Fraenkel set theory are valid in $V^{(H)}$. Accordingly the category $\mathcal{Set}^{(H)}$ of sets constructed within $V^{(H)}$ is a topos: in fact $\mathcal{Set}^{(H)}$ can be shown to be equivalent to the topos of canonical sheaves on *H.*
- There is a canonical embedding $x \mapsto \hat{x}$ of the universe V of sets into $V^{(H)}$ satisfying

$$[\![u \in \hat{x}]\!] = \bigvee_{y \in x} [\![u = \hat{y}]\!] \text{ for } x \in V, u \in V^{(H)}$$
$$x \in y \leftrightarrow V^{(H)} \models \hat{x} \in \hat{y}, \quad x = y \leftrightarrow V^{(H)} \models \hat{x} = \hat{y} \text{ for } x, y \in V$$
$$\varphi(x_1,\ldots,x_n) \leftrightarrow V^{(H)} \models \varphi(x_1,\ldots,x_n) \text{ for } x_1,\ldots,x_n \in V \text{ and restricted } \varphi$$

(Here a formula φ is *restricted* if all its quantifiers are restricted, i.e. can be put in the form $\forall x \in y$ or $\exists x \in y$.)

It follows from the last of these assertions that the canonical representative $\widehat{H}$ of H is a Heyting algebra in $V^{(H)}$. The *canonical prime filter* in $\widehat{H}$ is the H-set Φ_H defined by

$$dom(\Phi_H) = \{\hat{a} : a \in H\}, \quad \Phi_H(\hat{a}) = a \text{ for } a \in H.$$

Clearly $V^{(H)} \models \Phi_H \subseteq \widehat{H}$, and it is easily verified that

$V^{(H)} \models \Phi_H$ is a (proper) prime filter[3] in $\widehat{H}$.

It can also be shown that Φ_H is *V-generic* in the sense that, for any subset $A \subseteq H$,

[3] We recall that a filter F in a lattice is *prime* if $x \vee y \in F$ implies $x \in F$ or $y \in F$.

$$V^{(H)} \vDash \widehat{\bigvee A} \in \Phi_H \leftrightarrow \Phi_H \cap \hat{A} \neq \varnothing.$$

Moreover, for any $a \in H$ we have $[\![\hat{a} \in \Phi_H]\!] = a$, and in particular, for any sentence σ, $[\![\sigma]\!] = [\![\widehat{[\![\sigma]\!]} \in \Phi_H]\!]$. Thus $V^{(H)} \vDash \sigma \leftrightarrow V^{(H)} \vDash \widehat{[\![\sigma]\!]} \in \Phi_H$—in this sense Φ_H is the filter of "true" sentences in $V^{(H)}$.

This suggests that we define a *truth set* in $V^{(H)}$ to be an H-set F for which

$$V^{(H)} \vDash F \text{ is a filter in } \hat{H} \text{ such that } F \supseteq \Phi_H.$$

There is a natural bijective correspondence between truth sets in $V^{(H)}$ and weak nuclei on H. With each weak nucleus j on H we associate the H-set T_j defined by $\mathrm{dom}(T_j) = \mathrm{dom}(\Phi_H)$ and $T_j(\hat{a}) = j(a)$ for $a \in H$. It is easily verified that T_j is a truth set—the requirement that T_j be a filter corresponds exactly to the condition that j preserve finite meets and that it contain Φ_H to the condition that j satisfy $a \leq j(a)$. Inversely, given a truth set F in $V^{(H)}$, we define the map $j_F : H \to H$ by $j_F(a) = [\![\hat{a} \in T]\!]$. Again, it is readily verified that j_F is a weak nucleus on H. These correspondences are evidently mutually inverse and in fact establish an isomorphism between the frame $W(H)$ of weak nuclei on H and the internal frame of filters in $\hat{H}$ containing Φ_H. Under this isomorphism *nuclei* correspond precisely to *reflexive truth sets,* that is, truth sets satisfying the additional condition (evidently met by Φ_H)

$$V^{(H)} \vDash \widehat{[\![\hat{a} \in F]\!]} \in F \to \hat{a} \in F.$$

It is of interest to examine the familiar case in which H is a complete *Boolean algebra B.* In this case the canonical prime filter Φ_B is an *ultrafilter* in $\hat{B}$, so that, in $V^{(B)}$, the only filters in $\hat{B}$ containing Φ_B—the only truth sets—are Φ_B itself and $\hat{B}$. It follows that, for truth sets F and G in $V^{(B)}$

$$V^{(B)} \vDash F = G \leftrightarrow [\hat{0} \in F \leftrightarrow \hat{0} \in G].$$

In other words, the truth value $[\![\hat{0} \in F]\!]$, which can be an arbitrary member of B, determines the identity of F. This means that truth sets in $V^{(B)}$, and so equally weak nuclei on B, are in bijective correspondence with the members of B. In fact it is readily shown directly that any weak nucleus on a Boolean algebra B is of the form j_a for some $a \in B$. For given a weak nucleus j on B, observe: $\neg x \leq j(\neg x)$, whence $\neg j(\neg x) \leq \neg\neg x = x$. Also $j(x) \wedge j(\neg x) = j(x \wedge \neg x) = j(0)$, whence $j(x) \leq j(\neg x) \Rightarrow j(0) = \neg j(\neg x) \vee j(0)$ $\leq x \vee j(0)$. But clearly $x \vee j(0) \leq j(x)$, so that $j(x) = x \vee j(0)$.

Consider now the special case in which H is the completion $\widehat{P}$ of a preordered set P. We have already established a bijective correspondence between Grothendieck topologies on P and nuclei on $\widehat{P}$. This leads in turn to a bijective correspondence between Grothendieck topologies on P and reflexive truth sets in $V^{(\widehat{P})}$. Explicitly, this correspondence assigns to each Grothendieck topology $\mathbf{C}$ on P the reflexive truth set $T_{\mathbf{C}}$ in $V^{(\widehat{P})}$ given by $T_{\mathbf{C}}(S) = S^*$ for $S \in \widehat{P}$, and to each reflexive truth set F in $V^{(\widehat{P})}$ the Grothendieck topology $\mathbf{C}_F$ on P defined by $S \in \mathbf{C}_T(p) \leftrightarrow p \in [\![\hat{S} \in T]\!]$.

The topos $\mathcal{S}et^{(\widehat{P})}$ of sets in $V^{(\widehat{P})}$ is, as we have observed, equivalent to the topos of canonical sheaves on $\widehat{P}$, which is itself, as is well known, equivalent to the topos $\mathcal{S}et^{P^{op}}$ of presheaves on P. Moreover, Grothendieck topologies on P are known (see [4]) to correspond bijectively to internal Lawvere-Tierney topologies—that is, internal nuclei—on the truth-value object Ω in $\mathcal{S}et^{P^{op}}$. How this fact related to the representation of Grothendieck topologies as reflexive truth sets in $V^{(\widehat{P})}$? It turns out that in a general $V^{(H)}$ there is a natural bijection between truth sets/reflexive truth sets and weak nuclei/nuclei on $\Omega = \{u\colon u \subseteq \hat{1}\}$. The representation of Grothendieck topologies as truth sets in $V^{(H)}$, while equivalent to that through Lawvere-Tierney topologies, seems especially perspicuous.

The *forcing* relation $\Vdash_P$ in $V^{(\widehat{P})}$ between sentences and elements of P is defined by

$$p \Vdash_P \sigma \leftrightarrow p \in [\![\sigma]\!]^{\widehat{P}}.$$

Note that we then have

$$[\![\sigma]\!]^{\widehat{P}} = \{p : p \Vdash_P \sigma\}.$$

$\Vdash_P$ satisfies the usual rules governing Kripke semantics for predicate sentences, viz.,

- $p \Vdash_P \varphi \wedge \psi \leftrightarrow p \Vdash_P \varphi \;\; \& \;\; p \Vdash_P \psi$
- $p \Vdash_P \varphi \vee \psi \leftrightarrow p \Vdash_P \varphi$ or $p \Vdash_P \psi$
- $p \Vdash_P \varphi \rightarrow \psi \;\leftrightarrow\; \forall q \leq p[\, q \Vdash_P \varphi \;\rightarrow\; q \Vdash_P \psi]$
- $p \Vdash_P \neg\varphi \;\leftrightarrow\; \forall q \leq p \;\; q \nVdash_K \varphi$
- $p \Vdash_P \forall x\, \varphi \leftrightarrow p \Vdash_P \varphi(a)$ for every $a \in V^{(\widehat{P})}$
- $p \Vdash_P \exists x\, \varphi \leftrightarrow p \Vdash_P \varphi(a)$ for some $a \in V^{(\widehat{P})}$.

We note also that $\Vdash_P$ is *persistent* in the sense that, if $p \Vdash_P \varphi$ and $q \leq p$, then $q \Vdash_P \varphi$.

If **C** be a pretopology on P, the forcing relation $\Vdash_{\mathbf{C}}$ in the model $V^{(\widehat{\mathbf{C}})}$ is similarly defined by

$$p \Vdash_{\mathbf{C}} \sigma \leftrightarrow p \in [\![\sigma]\!]_{\widehat{\mathbf{C}}}.$$

As for Kripke models, this relation can be shown to satisfy the rules of Beth-Kripke-Joyal semantics, viz.,

- $p \Vdash_{\mathbf{C}} \varphi \wedge \psi \leftrightarrow p \Vdash_{\mathbf{C}} \varphi \;\; \& \;\; p \Vdash_{\mathbf{C}} \psi$
- $p \Vdash_{\mathbf{C}} \varphi \vee \psi \leftrightarrow \exists S \in \mathbf{C}(p)\; \forall s \in S\;\; [s \Vdash_{\mathbf{C}} \varphi$ or $s \Vdash_{\mathbf{C}} \psi]$
- $p \Vdash_{\mathbf{C}} \varphi \Rightarrow \psi \;\leftrightarrow\; \forall q \leq p[\, q \Vdash_{\mathbf{C}} \varphi \;\rightarrow\; q \Vdash_{\mathbf{C}} \psi]$
- $p \Vdash_{\mathbf{C}} \neg\varphi \;\leftrightarrow\; \forall q \leq p\;\; [q \Vdash_{\mathbf{C}} \varphi \rightarrow \varnothing \in \mathbf{C}(p)]$
- $p \Vdash_{\mathbf{C}} \forall x\varphi \leftrightarrow p \Vdash_{\mathbf{C}} \varphi(a)$ for every $a \in V^{(\widehat{\mathbf{C}})}$
- $p \Vdash_{\mathbf{C}} \exists x\varphi \leftrightarrow \exists S \in \mathbf{C}(p)\; \forall s \in S\;\; s \Vdash_{\mathbf{C}} \varphi(a)$ for some $a \in V^{(\widehat{\mathbf{C}})}$.

V. POTENTIAL APPLICATIONS OF COVER SCHEMES, KRIPKE MODELS, AND FRAME-VALUED SET THEORY IN SPACETIME PHYSICS

In spacetime physics any set $\mathscr{C}$ of events—a *causal set*—is taken to be partially ordered by the relation ≤ of *possible causation*: for $p, q \in \mathscr{C}$, $p \leq q$ means that *q* is in *p's* future light cone. In her groundbreaking paper [5] Fotini Markopoulou proposes that the causal structure of spacetime itself be represented by "sets evolving over $\mathscr{C}$" —that is, in essence, by the topos $\mathscr{Set}^{\mathscr{C}}$ of presheaves on $\mathscr{C}^{op.}$ To enable what she has done to be the more easily expressed within the framework presented here, we will reverse the causal ordering, that is, $\mathscr{C}$ will be replaced by $\mathscr{C}^{op}$, and the latter written as *P*—which will, moreover, be required to be no more than a *preordered* set. Thus *P* is a set of events preordered by the relation ≤, where $p \leq q$ is intended to mean that *p* is in *q's* future light cone—that *q could* be the cause of *p.* In requiring that ≤ be no more than a preordering—in dropping, that is, the antisymmetry of ≤—we are, in physical terms, allowing for the possibility that the universe is of Gödelian type, containing closed timelike lines.

Specifically, then, we fix a preordered set (*P*, ≤), which we shall call the *universal causal set*; its members will be called *events* and $p \leq q$ understood to mean that *p* is in *q*'s causal *future,* or *q*'s future light cone, in short, that *p* is a possible *effect* of *q*. (Thus, for each event *p,* the set $p{\downarrow}$ *is p*'s future light cone.) Markopoulou, in essence, suggests that viewing the universe "from the inside" amounts to placing oneself within the topos of presheaves or "evolving universe" $\mathscr{Set}^{P^{op}}$. Since, as we have already observed, $\mathscr{Set}^{P^{op}}$ is equivalent to the topos of sets in $V^{(\widehat{P})}$, Markopoulou's proposal may be effectively realized by working within

$V^{(\widehat{P})}$. Let us do so, writing for simplicity H for $\widehat{P}$: we think of $V^{(H)}$ as an *evolving universe*, and describing what the universe looks like "from the inside" will then amount to reporting the view from $V^{(H)}$. Each sentence σ of the language of set theory will be construed as an *assertion* concerning the evolving universe $V^{(H)}$.

The fact that each truth value $[\![\sigma]\!]^H$ (which we shall normally abbreviate to $[\![\sigma]\!]$) is a sieve in P— that is, satisfies $p \in [\![\sigma]\!]$ and $q \leq p \rightarrow q \in [\![\sigma]\!]$ may be understood as asserting that truth values in the evolving universe are "closed under potential effects", or "causally closed".

The forcing relation $\Vdash_P$ (which we will usually write simply as $\Vdash$) defined in the previous section now links events p and assertions σ: $p \Vdash \sigma$ will be taken to mean that σ *holds* as a result of (the occurrence of) event p, or that p *induces* the assertion σ to hold. The persistence of $\Vdash$ —i.e. the fact that, if $p \Vdash \sigma$ and $q \leq p$, then $q \Vdash \sigma$—amounts to the observation that, once an event p indeduces an assertion to hold, that assertion continues to hold throughout p's causal future[4].

Define the set $K \in V^{(H)}$ by dom(K) = $\{\hat{p} : p \in P\}$ and $K(\hat{p}) = p\downarrow$. Then, in $V^{(H)}$, K is a subset of $\widehat{P}$ and for $p \in P$, $[\![\hat{p} \in K]\!] = p\downarrow$. K is the counterpart in $V^{(H)}$ of the "evolving" set *Past* Markopoulou defines by *Past*(p) = $p\downarrow$, with insertions as transition maps. ($\widehat{P}$, incidentally, is the $V^{(H)}$- counterpart of the constant presheaf on P with value P which Markopoulou calls *World*.) Accordingly the "causal past" of any "event" p is represented by the truth value in $V^{(H)}$ of the statement $\hat{p} \in K$. The fact that, for any $p, q \in P$ we have

[4] It follows that assertions must be taken as being implicitly in the past tense: "such and such *was* the case".

$$q \Vdash_P \widehat{p} \in K \;\leftrightarrow\; q \le p$$

may be construed as asserting that *the events in the causal future of an event p are precisely those forcing (the canonical representative of) p to be a member of K.* For this reason we shall call *K* the *causal set in* $V^{(H)}$.

If we identify each $p \in P$ with $p{\downarrow} \in H$, *P* may then be regarded as a subset of *H* so that, in $V^{(H)}$, $\widehat{P}$ is a subset of $\widehat{H}$. It is not hard to show that, in $V^{(H)}$, *K* generates the canonical prime filter Φ_H in $\widehat{H}$. Using the *V*-genericity of Φ_H, and the density of *P* in *H*, one can show that $[\![\sigma]\!] = [\![\exists p \in K. p \le \widehat{[\![\sigma]\!]}]\!]$, so that, with moderate abuse of notation,

$$V^{(H)} \vDash [\sigma \leftrightarrow \exists p \in K.\ p \Vdash \sigma].$$

That is, in $V^{(H)}$, *a sentence holds precisely when it is forced to do so at some "causal past stage" in K.* This establishes the centrality of the causal set *K*—and, correspondingly, that of the "evolving" set *Past*— in determining the truth of sentences "from the inside", that is, inside the universe $V^{(H)}$.

Markopoulou also considers the complement of *Past.* In the present setting, this is the $V^{(H)}$-set $\neg K$—the *complement* of the causal set *K*—for which $[\![\widehat{p} \in \neg K]\!] = [\![p \notin K]\!] = \neg p{\downarrow}$. Markopoulou calls (*mutatis mutandis*) the events in $\neg p{\downarrow}$ those *beyond p's causal horizon*, in that no observer at *p* can ever receive "information" from any event in $\neg p{\downarrow}$. Since clearly we have

(★) $$q \Vdash \widehat{p} \in \neg K \;\leftrightarrow\; q \in \neg p{\downarrow},$$

it follows that *the events beyond the causal horizon of an event p are precisely those forcing (the canonical representative of) p to be a member of* $\neg K$. In this sense $\neg K$ reflects, or "measures" the causal structure of *P.*

In this connection it is natural to call $\neg\neg p{\downarrow}$ = $\{q : \forall r \le q \exists s \le r. s \le p]$ the *causal horizon* of *p*: it consists of those events *q* for which an observer placed at *p* could, in its future, receive information from any event in the future of an observer placed at *q*. Since

$$q \Vdash \hat{p} \in \neg\neg K \;\leftrightarrow\; q \in \neg\neg p{\downarrow},$$

it follows that *the events within the causal horizon of an event are precisely those forcing (the canonical representative of) p to be a member of* $\neg\neg K$.

It is easily shown that $\neg K$ is *empty* (i.e. $V^{(H)} \models \neg K = \varnothing$) if and only if *P* is *directed downwards* in the sense that for any $p, q \in P$ there is $r \in P$ for which $r \leq p$ and $r \leq q$; that is, if *the future light cones of any pair of events have nonempty intersection or "overlap".* This holds in the case, considered by Markopoulou, of *discrete Newtonian time evolution*—in the present setting, the case in which *P* is the opposite $\mathbb{N}^{op}$ of the totally ordered set $\mathbb{N}$ of natural numbers. Here the corresponding complete Heyting algebra *H* is the family of all downward-closed sets of natural numbers. Interestingly, in this case, the causal set *K is neither finite nor actually infinite.*

To see this, first note that, for any natural number *n,* we have, $[\![\neg(\hat{n} \in \neg K)]\!] = \mathbb{N}$. It follows that $V^{(H)} \models \neg\neg\forall n \in \hat{\mathbb{N}}.\ n \in K$. But, working in $V^{(H)}$, if $\forall n \in \hat{\mathbb{N}}.\ n \in K$, then *K* is not finite, so if *K* is finite, then $\neg\forall n \in \hat{\mathbb{N}}.\ n \in K$, and so $\neg\neg\forall n \in \hat{\mathbb{N}}.\ n \in K$ implies the non-finiteness of *K.*

But, in $V^{(H)}$, *K* is not actually infinite. For (again working in $V^{(H)}$), if *K* were actually infinite (i.e., if there existed an injection of $\hat{\mathbb{N}}$ into *K*), then the statement

$$\forall x \in K\, \exists y \in K.\ x > y$$

would also have to hold in $V^{(H)}$. But calculating that truth value gives:

$$\begin{aligned}[\![\forall x \in K \exists y \in K . x > y]\!] &= \bigcap_{m\in\mathbb{N}^{op}} [m \downarrow \Rightarrow \bigcup_{n\in\mathbb{N}^{op}} n \downarrow \cap [\![\hat{m} > \hat{n}]\!]] \\ &= \bigcap_{m} [m \downarrow \Rightarrow \bigcup_{n<m} n \downarrow] \\ &= \bigcap_{m} [m \downarrow \Rightarrow (m+1) \downarrow] \\ &= \bigcap_{m} (m+1) \downarrow \\ &= \varnothing\end{aligned}$$

So $\forall x \in K\ \exists y \in K.\ x > y$ is false in $V^{(H)}$ and therefore K is not actually infinite.

In other words, in evolving Newtonian spacetime, the set K representing past time is potentially, but not actually infinite: this is, in essence, what Kant asserted of time.

In order to formulate an observable causal *quantum theory* Markopoulou considers the possibility of introducing a *causally evolving algebra of observables.* This amounts to specifying a presheaf of C^*-algebras on P, which, in the present framework, corresponds to specifying a set $\mathscr{A}$ in $V^{(H)}$ satisfying

$$V^{(H)} \vDash \mathscr{A} \text{ is a } C^*\text{-algebra.}$$

The "internal" C^*-algebra $\mathscr{A}$ is then subject to the intuitionistic internal logic of $V^{(H)}$: *any* theorem concerning C^*-algebras—provided only that it be constructively proved—automatically applies to $\mathscr{A}$. Reasoning with $\mathscr{A}$ is more direct and simpler than reasoning with $\mathcal{A}$.

This same procedure of "internalization" can be performed with any causally evolving object: each such object of type $\mathscr{T}$ corresponds to a set S in $V^{(H)}$ satisfying

$$V^{(H)} \vDash S \text{ is of type } \mathscr{T}.$$

Internalization may also be applied in the case of the presheaves *Antichains* and *Graphs* considered by Markopoulou. Here, for each event *p, Antichains*(*p*) consists of all sets of causally unrelated events in *Past*(*p*), while *Graphs*(*p*) is the set of all graphs supported by elements of *Antichains*(*p*). In the present framework *Antichains* is represented by the $V^{(H)}$–set $Anti = \{ X \subseteq \widehat{P} : X \text{ is an antichain}\}$ and *Graphs* by the $V^{(H)}$–set *Grph* $= \{G: \exists X \in A \,.\, G \text{ is a graph supported by } A\}$. Again, both *Anti* and *Grph* can be readily handled using the internal intuitionistic logic of $V^{(H)}$.

Finally let us examine the role of cover schemes on causal sets. Suppose we are given a cover scheme **C** on the universal causal set *P*. Each **C-**cover of an event *p* may be thought of as a "sampling" of the events in *p*'s causal future, a "survey" of *p*'s potential effects—in a word, a *survey of p.* Using this language the defining condition **(Cov)** for cover schemes laid down in section I becomes: *for any survey S of a given event p, and any event q which is a possible effect of p, there exists a survey of q each event in which is the possible effect of some event in S.*

As we have seen, cover schemes may be used to force certain conditions to prevail in the associated models. Let us consider, for example, the cover scheme **Den** in *P.* We know that the associated frame $\widehat{\mathbf{Den}}$ is a Boolean algebra—let us denote it by *B.* The corresponding causal set K_B in $V^{(B)}$ then has the property

$$[\![\widehat{p} \in K_B]\!] = \neg\neg p \downarrow;$$

so that,

$$q \Vdash_B \widehat{p} \in K_B \leftrightarrow q \in \neg\neg p\downarrow$$

$$\leftrightarrow \; q \text{ is in } p\text{'s causal horizon.}$$

Comparing this with **(★)** above, we see that moving to the universe $V^{(B)}$—"Booleanizing" it, so to speak—*amounts to replacing causal futures by causal horizons.* When *P* is linearly ordered, as for example in the case of Newtonian time, the causal horizon of any event coincides with the whole of *P*, *B* is the two-element Boolean algebra **2,** so that $V^{(B)}$ is just the universe V of "static" sets. In this case, then, the effect of "Booleanization" is to *render the universe timeless.*

The universes associated with the cover schemes $\mathbf{C}^A$ and $\mathbf{C}_A$ seem also to have a rather natural physical meaning. Consider, for instance the case in which A is the sieve $p{\downarrow}$—the causal future of p. In the associated universe $V^{(\widehat{\mathbf{C}^A})}$ the corresponding causal set K^A satisfies

$$[\![\hat{q} \in K^A]\!] = \textit{least } \mathbf{C^A}\textit{-closed sieve containing } q$$

so that , in particular

$$[\![\hat{p} \in K^A]\!] = \textit{least } \mathbf{C^A}\textit{-closed sieve containing } p$$

$$= P.$$

This means that, for every event q,

$$q \Vdash_{\widehat{\mathbf{C^A}}} \hat{p} \in K^A.$$

Comparing this with **(★)**, we see that in $V^{(\widehat{\mathbf{C}^A})}$ that every event has been "forced" into p's causal future: in short, that p now marks the "beginning" of the universe as viewed from inside $V^{(\widehat{\mathbf{C}^A})}$.

Similarly, we find that the causal set K_A in the universe $V^{(\widehat{\mathbf{C}_A})}$ has the property

$$q \le p \to \forall r[r \Vdash_{\widehat{\mathbf{C_A}}} \hat{q} \in \neg K_A].$$

a comparison with **(★)** above reveals that, in $V^{(\widehat{\mathbf{C}^A})}$, every event—including p itself—has been placed beyond p's causal horizon. In effect, the event p has been obliterated, effaced from the universe—like the extraordinary events in H.G. Wells's *The Man Who Could Work Miracles,* the event p never occurred!

As a final possibility consider the universe $V^{(\widehat{\tilde{P}})}$ associated with the free lower semilattice $\tilde{P}$ generated by P. In this case the elements of $\tilde{P}$ are finite sets of events, preordered by the relation $\sqsubseteq$: for F, $G \in \tilde{P}$, $F \sqsubseteq G$ iff every event in G is in the causal past of an event in F. The empty set of events is the top element of $\widetilde{P}$. The causal set $\widetilde{K}$ in $V^{(\widetilde{P})}$ has the property that its complement $\neg\widetilde{K}$ is empty (so that, in this universe, the light cones of any pair of "events" overlap) and $\widehat{\varnothing}$ is an initial event in the sense that $F \Vdash_{\widetilde{P}} \widehat{\varnothing} \in \widetilde{K}$ for every "event" F. In this case passage to the new universe $V^{(\widetilde{P})}$ preserves the original causal relations in the sense that

$$\{q\} \Vdash_{\widetilde{P}} \widehat{\{p\}} \in \widetilde{K} \leftrightarrow q \Vdash_{P} \widehat{p} \in K .$$

In other words, in passing to the new universe the initial event $\widehat{\varnothing}$ and the new light cone overlaps have been "freely adjoined" to the original universe.

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